Extrapolation in Time of Fields with Stationary Increments as Applied to the Impact Point Prediction of Free Rockets

FERDINAND P. BEER* AND ROBERT G. SARUBBI†
Lehigh University, Bethlehem, Pa.

The approach of Kolmogoroff and Yaglom is used to characterize nonstationary random processes possessing stationary increments. Structure functions are introduced and used to extend Wiener's extrapolation theory to this class of processes. The results obtained are applied to the prediction of the angle at burnout of a free (unguided) rocket from continuous wind-velocity measurements taken at various elevations and interrupted a short time before the firing of the rocket. The mean square of the error involved is shown to depend upon the cross-structure function of the wind-velocity field. The proposed method does not require the identification and elimination of the linear trend usually present in such a field.

I. Introduction

A BASIC premise of probability theory is that a continuous random time series $f(t)$ is determined statistically by the complete system of joint probability distribution functions at any n values of t. When the distribution functions depend only upon time differences, we term the process stationary and usually content ourselves with a determination of first and second moments, i.e., the mean and the correlation function instead of the complete system of joint probability distribution functions. If, in addition, the process is Gaussian, the first and second moments are sufficient to specify the statistics of the process.

In practice, a process is usually assumed stationary, and ensemble averages and time averages taken with respect to one realization of the process are assumed equivalent. These two assumptions simplify otherwise long and detailed data gathering and processing techniques that would attempt to use a representative ensemble for analysis of the process.

When the process is clearly nonstationary, one usually resorts to either of the following approaches: 1) collect a truly representative ensemble and obtain the various time-dependent moments of the process by ensemble averaging, or 2) assume that the process is the combination of a stationary process and a time-dependent polynomial.

The first of these approaches is explored extensively by Bendat[1] in terms of two-dimensional spectra. If there is any objection to this approach, it is that, from a practical point of view, only a single record or at most a few representative sample functions of the ensemble are usually available.

The second approach presupposes a specific kind of random process and requires the determination of the time-dependent polynomial that is to be subtracted out before operations on the stationary portion of the process can be made. It may hardly be expected that that part of the statistics of any process that is time-dependent can be determined without reference to the ensemble.

Another characterization of a nonstationary process appears in Wiener[2] in connection with the problem of extrapolation, where processes with stationary nth derivatives are considered.

In a paper by Yaglom[3] (see also Ref. 4), a detailed mathematical treatment of a broader class of processes called "processes with random stationary nth increments" is given. The technique, originally introduced by Kolmogoroff in studies of turbulence, depends upon removing large-scale, long-range fluctuations in the process by forming successive differences, until the new process thus obtained is stationary. In Ref. 5, Tatarskii suggests that this technique be used as a statistical base for describing the various meteorological variables of the atmosphere.

Our purpose here is 1) to outline the essential features of processes with stationary increments, 2) to extend the classical extrapolation methods to processes with stationary increments, and 3) to show how the results obtained may be used in the prediction of the impact point of free rockets.

II. Processes with Stationary Increments

Consider the function $f(t)$ and denote by $\Delta_\nu f(t)$ the increment $f(t) - f(t-\nu)$. The increment of the first increment, denoted by $\Delta_\nu^2 f(t)$, is

$$\Delta_\nu^2 f(t) = \Delta_\nu[\Delta_\nu f(t)] = f(t) - 2f(t-\nu) + f(t-2\nu) \tag{1}$$

Proceeding in this way, one can eventually find the expression for the nth order increment

$$\Delta_\nu^n f(t) = \sum_{l=0}^{n} (-1)^l \frac{n!}{l!(n-l)!} f(t - l\nu) \tag{2}$$

Equation (2) possesses the essential feature of a difference approximation to the nth derivative.

A random nth increment is called stationary in the "weak sense" if the expectations

$$C_f{}^n(\nu) = \langle \Delta_\nu{}^n f(t)\rangle \tag{3}$$

$$D_{ff}{}^n(\tau,\nu) = \langle \Delta_\nu{}^n f(t)\Delta_\nu{}^n f(t+\tau)\rangle \tag{4}$$

are independent of t. The brackets stand for averages taken over the ensemble but, in applications, will be replaced by time averages. The function $D_{ff}{}^n(\tau,\nu)$ is the correlation function of the nth increment and is called the structure function of the process. Yaglom[3] shows that the functional form of Eq. (3) must be

$$C_f{}^n(\nu) = \langle \Delta_\nu{}^n f(t)\rangle = c\nu^n \tag{5}$$

A generalization of the structure function for two processes $f(t)$ and $g(t)$ can be made by cross-correlating the stationary nth differences. The result, denoted by $D_{fg}{}^n(\tau,\nu)$, is called the cross-structure function:

$$D_{fg}{}^n(\tau,\nu) = \langle \Delta_\nu{}^n f(t)\Delta_\nu{}^n g(t+\tau)\rangle \tag{6}$$

Received June 3, 1964. This paper is the result of research sponsored by the U. S. Army Signal Corps under Grant No. DA-SIG-36-039-61-G7.

* Professor of Mechanics.

† Assistant Professor of Mechanics.

Since the structure function $D_{ff^n}(\tau,\nu)$ is actually the correlation function of the stationary process consisting of the increments, it may be represented as the Fourier transform of a function $S_{ff^n}(\omega,\nu)$:

$$D_{ff^n}(\tau,\nu) = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_{ff^n}(\omega,\nu)e^{i\omega\tau}d\omega \tag{7}$$

The inverse relation is

$$S_{ff^n}(\omega,\nu) = \int_{-\infty}^{\infty} D_{ff^n}(\tau,\nu)e^{-i\omega\tau}d\tau \tag{8}$$

In practice, the function $D_{ff^n}(\tau,\nu)$ is constructed from the available data by means of time averages and the spectral density $S_{ff^n}(\omega,\nu)$ is determined from Eq. (8). As in the case of the computation of the spectral density of any stationary process, the mean value of the increments determined from a time average over each record must be subtracted out before the calculation of the structure function (4) is made; the function $S_{ff^n}(\omega,\nu)$ will thus actually represent the spectral density of the process $\Delta_\nu{}^nf(t) - c\nu^n$, which has a zero mean.

It may be shown[3] that theoretically the spectral density can be written in the form

$$S_{ff^n}(\omega,\nu) = 2^n(1 - \cos\omega\nu)^nS_{ff}(\omega) \tag{9}$$

The function $S_{ff}(\omega)$, called the spectrum of the process, is independent of the time interval ν. Relations similar to (8) and (9), which involve the cross-structure function $D_{fg^n}(\tau,\nu)$ and the corresponding cross-spectrum $S_{fg}(\omega)$ of the processes $f(t)$ and $g(t)$, can be written.

Now let us examine the first-order structure function $D_{ff}(\tau,\nu)$ for the particular case in which the process $f(t)$ is stationary. Making $n = 1$ in Eq. (4), we have

$$\begin{aligned} D_{ff}(\tau,\nu) &= \langle[f(t) - f(t-\nu)][f(t+\tau) - f(t+\tau-\nu)]\rangle \\ &= \langle f(t)f(t+\tau)\rangle + \langle f(t-\nu)f(t+\tau-\nu)\rangle \\ &\quad - \langle f(t-\nu)f(t+\tau)\rangle - \langle f(t)f(t+\tau-\nu)\rangle \end{aligned}$$

Since the process is stationary, the averages obtained are independent of t and represent values of the correlation function φ_{ff} of the process. We write, therefore,

$$D_{ff}(\tau,\nu) = 2\varphi_{ff}(\tau) - \varphi_{ff}(\tau+\nu) - \varphi_{ff}(\tau-\nu) \tag{10}$$

or, expressing the correlation function as the inverse Fourier transform of the power spectrum $\Phi_{ff}(\omega)$,

$$\begin{aligned} D_{ff}(\tau,\nu) &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\Phi_{ff}(\omega)[2e^{i\omega\tau} - e^{i\omega(\tau+\nu)} - e^{i\omega(\tau-\nu)}]d\omega \\ D_{ff}(\tau,\nu) &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\Phi_{ff}(\omega)\, 2(1 - \cos\omega\nu)e^{i\omega\tau}d\omega \end{aligned} \tag{11}$$

Comparing Eqs. (7, 9, and 11), we see that the function $S_{ff}(\omega)$ represents the power spectrum $\Phi_{ff}(\omega)$ of the process when the process is stationary. We also note from Eq. (10) that, when the process is stationary, the structure function $D_{ff}(\tau,\nu)$ is bounded and that it approaches $2\varphi_{ff}(\tau)$ as ν increases(indefinitely.

In order to show the relation between the spectrum of a process with stationary first increments, and the spectrum of the derivative of the process, we write Eq. (7) for $n = 1$ in the more general form[3]

$$D_{ff}(\tau;\nu_1,\nu_2) = \frac{1}{2\pi}\int_{-\infty}^{\infty}(1 - e^{i\omega\nu_1})(1 - e^{-i\omega\nu_2})\, S_{ff}(\omega)e^{i\omega\tau}d\omega \tag{12}$$

Dividing both members of (12) by the product of ν_1 and ν_2 and letting both of these go to zero, we obtain the relation between the correlation function of the derivative $\dot{f}(t)$ of $f(t)$ and the spectrum $S_{ff}(\omega)$ of the process:

$$\varphi_{\dot{f}\dot{f}}(\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\omega^2S_{ff}(\omega)e^{i\omega\tau}d\omega \tag{13}$$

Equation (13) shows that $\omega^2S_{ff}(\omega)$ is the power spectrum of the derivative of a process that possesses stationary first increments. If the process is not itself stationary, the function $S_{ff}(\omega)$ has a singularity of the type ω^{-2}.

III. Extrapolation of a Nonstationary Process with Stationary Increments

Let us determine how best to operate linearly on a nonstationary process known for $-\infty < t < t_1$, to obtain an estimate of the process at a time $t_1 + \nu$. We shall assume stationary first increments and apply the Wiener prediction theory, using the technique of Bode and Shannon,[6] for the extrapolation of these increments.

Consider a random process $f(t)$ with stationary first increments $\Delta_\nu f(t)$ that possess a rational continuous power-spectral density $S(\omega,\nu)$. The filter that produces a process of spectral density $S(\omega,\nu)$ when subjected to a white-noise input $w(t)$ has the transfer function $K_\nu(\omega)$ defined by

$$|K_\nu(\omega)|^2 = S(\omega,\nu) \tag{14}$$

If the filter is to be physically realizable, we must construct $K_\nu(\omega)$ with the factors of $S(\omega,\nu)$ which have poles in the upper half of the complex ω plane.

The impulse response function corresponding to $K_\nu(\omega)$ is

$$k_\nu(\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty}K_\nu(\omega)e^{i\omega\tau}d\omega \tag{15}$$

and the increment may be expressed at time t_1 as

$$\Delta_\nu f(t_1) = \int_0^\infty k_\nu(\tau)w(t_1 - \tau)d\tau$$

Similarly, we have at time $t_1 + \nu$

$$\Delta_\nu f(t_1+\nu) = \int_0^\infty k_\nu(\tau)w(t_1 + \nu - \tau)d\tau \tag{16}$$

The process $w(t)$ is not known beyond the time t_1; thus, the lower limit must be raised from 0 to ν, yielding the predicted value of the increment:

$$\Delta_\nu{}^pf(t_1+\nu) = \int_\nu^\infty k_\nu(\tau)w(t_1 + \nu - \tau)d\tau \tag{17}$$

or

$$\Delta_\nu{}^pf(t_1+\nu) = \int_0^\infty k_\nu(\tau+\nu)w(t_1 - \tau)d\tau \tag{18}$$

Setting

$$\begin{aligned} g_\nu(\tau) &= k_\nu(\tau+\nu) \qquad (\tau > 0) \\ g_\nu(\tau) &= 0 \qquad (\tau < 0) \end{aligned} \tag{19}$$

the extrapolated value of the increment is expressed as the response to the white noise input $w(t)$ of a filter with impulse response function $g_\nu(\tau)$:

$$\Delta_\nu{}^pf(t+\nu) = \int_0^\infty g_\nu(\tau)w(t_1 - \tau)d\tau \tag{20}$$

The transfer function corresponding to $g_\nu(\tau)$ is

$$G_\nu(\omega) = \int_{-\infty}^{\infty} g_\nu(\tau)e^{-i\omega\tau}d\tau$$

or, recalling (15) and (19),

$$\begin{aligned} G_\nu(\omega) &= \int_0^\infty k_\nu(\eta+\nu)e^{-i\omega\eta}d\eta \\ G_\nu(\omega) &= \frac{1}{2\pi}\int_0^\infty e^{-i\omega\eta}d\eta\int_{-\infty}^{\infty}K_\nu(\Omega)e^{i\Omega(\eta+\nu)}d\Omega \end{aligned} \tag{21}$$

Noting that the white noise $w(t)$ used in Eq. (20) is itself obtained by passing the process $\Delta_\nu f(t)$ through a filter of transfer function $K_\nu^{-1}(\omega)$, we conclude that the extrapolated value $\Delta_\nu^p f(t_1 + \nu)$ of the process may be obtained by passing the available record $\Delta_\nu f(t)$, $-\infty < t < t_1$, through a filter of transfer function

$$P_\nu(\omega) = G_\nu(\omega)/K_\nu(\omega)$$

where $K_\nu(\omega)$ and $G_\nu(\omega)$ are defined in (14) and (21), respectively. Expressed in terms of $K_\nu(\omega)$, the transfer function $P_\nu(\omega)$ of the predictor is

$$P_\nu(\omega) = \frac{1}{2\pi K_\nu(\omega)} \int_0^\infty e^{-i\omega\eta}\, d\eta \int_{-\infty}^{\infty} K_\nu(\Omega) e^{i\Omega(\eta+\nu)} d\Omega \quad (22)$$

In calculating the spectrum $S(\omega,\nu)$ that led to the predictor $P_\nu(\omega)$, the mean value $c\nu$ of the process $\Delta_\nu f(t)$ was subtracted out. Therefore, to predict the value $\Delta_\nu^p f(t_1 + \nu)$, it is necessary to operate with $P_\nu(\omega)$ on the process $\Delta_\nu f(t) - c\nu$ and to add $c\nu$ to the result obtained. Introducing the impulse response function

$$p_\nu(\tau) = \frac{1}{2\pi} \int_{-\infty}^{\infty} P_\nu(\omega) e^{i\omega\tau} d\omega \quad (23)$$

of the predictor, we express the extrapolated value of the increment $\Delta_\nu f(t)$ at time $t_1 + \nu$ as follows:

$$\Delta_\nu^p f(t_1 + \nu) = c\nu + \int_0^\infty p_\nu(\tau)[\Delta_\nu f(t_1 - \tau) - c\nu] d\tau \quad (24)$$

Thus, the extrapolated value $\Delta_\nu^p f(t_1 + \nu)$ may be determined from an average of the values of the increment process for $t \leq t_1$, using the weighting function $p_\nu(\tau)$.

Once the extrapolated value of the increment has been determined, the predicted value $f^p(t_1 + \nu)$ of the process itself is readily obtained. We have

$$f^p(t_1 + \nu) = f(t_1) + \Delta_\nu^p f(t_1 + \nu) \quad (25)$$

The error in the predicted value of the process is obtained by subtracting (16) from (17):

$$e = \Delta_\nu^p f(t_1 + \nu) - \Delta_\nu f(t_1 + \nu) = -\int_0^\nu k_\nu(\tau) w(t_1 + \nu - \tau) d\tau \quad (26)$$

Since the white noise $w(t)$ has zero mean, $\langle e \rangle$ is zero; thus $\Delta_\nu^p f(t_1 + \nu)$ is the prediction that corresponds to the minimum value of the mean square $\langle e^2 \rangle$ of the error.[6]

Referring to Eq. (9), we note that, if the time interval ν is not sufficiently small, the power-spectral density $S(\omega,\nu)$ of the increment $\Delta_\nu f(t)$ will include several zeros along the ω axis within the interval over which $S(\omega,\nu)$ is to be expressed as a rational function of ω. Also, the process $\Delta_\nu f(t)$ may not be truly stationary. To avoid both of these difficulties, it may be desirable to use increments $\Delta_\nu f(t)$ corresponding to a time interval ν smaller than the interval μ defining the time $t_1 + \mu$ at which the process $f(t)$ is to be predicted. Let $P_{\mu/\nu}(\omega)$ be the transfer function of the predictor with which we should operate on the record $\Delta_\nu f(t)$, $t \leq t_1$, to obtain the predicted value $\Delta_\mu^p f(t_1 + \mu)$ of the increment $\Delta_\mu^p f(t)$. This transfer function is obtained by substituting $G_\mu(\omega)$ for $G_\nu(\omega)$ in our earlier derivation, i.e., by replacing $K_\nu(\Omega)$ in Eq. (22) by $K_\mu(\Omega)$ and ν by μ in the exponential function in the same equation. But, referring to Eqs. (9) and (14), we observe that

$$K_\mu(\Omega) = K_\nu(\Omega)\,(1 - e^{-\Omega\mu})/(1 - e^{-i\Omega\nu}) \quad (27)$$

Thus, the transfer function of the desired predictor is

$$P_{\mu/\nu}(\omega) = \frac{1}{2\pi K_\nu(\omega)} \int_0^\infty e^{-i\omega\eta}\, d\eta \int_{-\infty}^{\infty} K_\nu(\Omega) \times \frac{1 - e^{-i\Omega\mu}}{1 - e^{-i\Omega\nu}} e^{i\Omega(\eta+\mu)}\, d\Omega \quad (28)$$

The extrapolated value of the increment $\Delta_\mu f(t)$ at time $t_1 + \mu$ is obtained by replacing the weighting function $p_\nu(\tau)$ in (24) by the impulse response function $p_{\mu/\nu}(\tau)$ of the predictor $P_{\mu/\nu}(\omega)$. The predicted value of the process $f(t)$ itself is

$$f^p(t_1 + \mu) = f(t_1) + \Delta_\mu^p f(t_1 + \mu) \quad (29)$$

The results obtained in this section may be extended to higher-order increments. For instance, to predict the value $(\Delta_\nu^2)^p f(t)$ of the second increment of the process $f(t)$, we can construct a predictor from the power-spectral density $S^2(\omega,\nu)$ of the second increments, and operate with this predictor on the process $\Delta_\nu^2 f(t) - c\nu^2$. The mean value $c\nu^2$ of the increments is then added to the result obtained. Referring to Eq. (1), we verify that the predicted value of the process $f(t)$ may then be expressed as

$$f^p(t_1 + \nu) = 2f(t_1) - f(t_1 - \nu) + (\Delta_\nu^2)^p f(t_1 + \nu) \quad (30)$$

IV. Application to the Prediction of the Impact Point of Free Rockets

Consider an unguided rocket fired vertically. Since the rocket is disturbed by the wind field it encounters as it rises, the tangent to its trajectory at burnout will form an angle q with the vertical. This angle may be determined for any given rocket from the wind profile at the time of firing, and the impact point of the rocket may, in turn, be determined from q.

We shall assume that, before each firing, wind data is available for a large number of elevations z in the form of velocity records $u(t,z)$ continuous in time. Such information may be provided (at least in the lower part of the atmosphere, where the rocket is most sensitive to wind disturbances) by anemometers mounted on a tower. Since the computation of the angle at burnout q from the wind velocity data takes a certain amount of time, the reading of the records must stop at a time prior to the firing. Let t_1 be that time, and let $t_1 + \mu$ be the time at which the rocket actually encounters the wind field. Although the time lag μ depends upon the elevation considered, it will be assumed here for simplicity that the time of ascent of the rocket is negligible and, thus, that $t_1 + \mu$ is equal to the firing time.

The wind velocity $u(t_1 + \mu,z)$ at firing time will be predicted at each level z from the continuous record $u(t,z)$, $t \leqq t_1$, obtained at that level. Although wind velocity is not a stationary random process, it may be considered over a reasonably long time interval to be a random process with stationary first increments.[5,7] The method developed in the preceding section and leading to Eqs. (28) and (29) may therefore be used. The first increments $\Delta_\nu u(t,z)$ will be formed at each level from the corresponding wind-velocity record and will be subjected to the predictor $P_{\mu/\nu}(\omega,z)$. This predictor will have been determined before hand by considering a number of wind-velocity records taken at the level z, and computing the corresponding structure function $D(\tau,\nu;z,z)$ and spectral density $S(\omega,\nu;z,z)$. Since, as indicated earlier, the mean value $c(z)\nu$ of the increment $\Delta_\nu u(t,z)$ must be subtracted out before any record is used, the spectral density $S(\omega,\nu;z,z)$ and the resulting predictor $P_{\mu/\nu}(\omega,z)$ will be independent of any linear trend contained in the velocity records. Referring to Eq. (28), we note that the predictor will also be independent of the mean square of $\Delta_\nu u(t,z)$. It may therefore be expected that the predictors $P_{\mu/\nu}(\omega,z)$ and the corresponding weighting functions $p_{\mu/\nu}(\tau,z)$ determined at the various levels z will remain valid under very general atmospheric conditions. It should be noted, on the other hand, that the mean value $c(z)\nu$ of the increment must be determined at each level prior to each firing; the proportional quantity $c(z)\mu$ may then be added to the output of each predictor $P_{\mu/\nu}(\omega,z)$, and the correct values of the predicted increment $\Delta_\mu^p u(t_1 + \mu,z)$ and of the predicted wind velocity

$$u^p(t_1 + \mu,z) = u(t,z) + \Delta_\mu^p u(t_1 + \mu,z) \quad (31)$$

will be obtained at each level z.

The predicted value $q^p(t_1 + \mu)$ of the angle at burnout and the corresponding impact point will be determined from the predicted wind-velocity profile (31) by integrating the appropriate equations of motion. Assuming that the trajectory of the rocket is contained in a fixed vertical plane and that its motion may be described, at least approximately, by linear differential equations, we shall now write the difference between the value $q^p(t_1 + \mu)$ of the angle at burnout obtained from the extrapolated wind-velocity profile and the value $q(t_1)$ which would be obtained from the profile read at time t_1. We have

$$q^p(t_1 + \mu) - q(t_1) = \int_0^{z_B} h(z,z_B)[u^p(t_1 + \mu,z) - u(t_1,z)]dz$$

or

$$q^p(t_1 + \mu) - q(t_1) = \int_0^{z_B} h(z,z_B)\Delta_\mu{}^p u(t_1 + \mu,z)dz \quad (32)$$

where $h(z,z_B)$ represents the response of the rocket at burnout elevation z_B to a unit wind impulse occurring at elevation z.

V. Determination of the Error in Prediction

The difference between the actual value $q(t_1 + \mu)$ of the angle at burnout, computed from the wind-velocity profile the rocket will encounter at time $t_1 + \mu$, and the value $q(t_1)$ computed from the profile at time t_1, is

$$q(t_1 + \mu) - q(t_1) = \int_0^{z_B} h(z,z_B)\Delta_\mu u(t_1 + \mu,z)dz \quad (33)$$

Subtracting (33) from (32), we obtain the error e in the prediction of the angle at burnout:

$$e = q^p(t_1 + \mu) - q(t_1 + \mu) = \int_0^{z_B} h(z,z_B)[\Delta_\mu{}^p u(t_1 + \mu,z) - \Delta_\mu u(t_1 + \mu,z)]dz \quad (34)$$

Considering first the particular case when $\mu = \nu$, and assuming that the mean value has been subtracted from the increment process, we represent the extrapolated value $\Delta_\nu{}^p u(t_1 + \nu,z)$ by the convolution

$$\Delta_\nu{}^p u(t_1 + \nu,z) = \int_0^\infty p_\nu(\eta,z)\Delta_\nu u(t_1 - \eta,z)d\eta \quad (35)$$

where $p_\nu(\eta,z)$ denotes the impulse response function corresponding to the predictor $P_\nu(\omega,z)$. On the other hand, we have

$$\Delta_\nu u(t_1 + \nu,z) = \int_{-\infty}^{\infty} \delta(\eta + \nu)\Delta_\nu u(t_1 - \eta,z)d\eta \quad (36)$$

where $\delta(\eta)$ is the Dirac delta function. Substituting from (35) and (36) into (34), where μ has been replaced by ν, and noting that the lower limit in (35) may be replaced by $-\infty$ since $p_\nu(\eta,z)$ is zero for $\eta < 0$, we write

$$e = \int_0^{z_B} h(z,z_B)dz \int_{-\infty}^{\infty} [p_\nu(\eta,z) - \delta(\eta + \nu)]\Delta_\nu u(t_1 - \eta,z)d\eta \quad (37)$$

Squaring both members of (37) and averaging, we obtain the following expression for the mean square of the error:

$$\langle e^2\rangle = \int_0^{z_B}\int_0^{z_B} h(z,z_B)\, h(z',z_B)\, F(z,z')dzdz' \quad (38)$$

where

$$F(z,z') = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} [p_\nu(\eta,z) - \delta(\eta + \nu)][p(\eta',z') - \delta(\eta' + \nu)]\,\langle\Delta_\nu u(t_1 - \eta,z)\Delta_\nu u(t_1 - \eta',z')\rangle d\eta d\eta' \quad (39)$$

We note that the average in the integrand of (39) represents the cross-structure function

$$D(\eta - \eta',\nu;z,z') = \langle\Delta_\nu u(t - \eta,z)\Delta_\nu u(t - \eta',z')\rangle \quad (40)$$

and that it may be obtained by forming time averages of the product of the increments at levels z and z' for each realization of the field, and averaging these values over all available records of the ensemble. From the cross-structure function, we may compute the cross-power spectral density of the increment process

$$S(\omega,\nu;z,z') = \int_{-\infty}^{\infty} D(\tau,\nu;z,z')e^{-i\omega\tau}\, d\tau \quad (41)$$

Substituting the inverse Fourier transform of (41) into Eq. (39), and observing that the Fourier transform of $\delta(\eta + \nu)$ is $e^{i\omega\nu}$, we write

$$F(z,z') = \frac{1}{2\pi}\int_{-\infty}^{\infty} [P_\nu{}^*(\omega,z) - e^{-i\omega\nu}] \times [P_\nu(\omega,z') - e^{i\omega\nu}]S(\omega,\nu;z,z')d\omega \quad (42)$$

Considering now the more general case when the firing time lag μ is different from the time interval ν for which the first increments have been computed, we replace $P_\nu(\omega,z)$ in Eq. (42) by $P_{\mu/\nu}(\omega,z)$ and multiply the new delay transfer function $e^{i\omega\mu}$ by the transfer function of the filter which transforms $\Delta_\nu u(t)$ into $\Delta_\mu u(t)$. Observing from Eq. (27) that this last function is equal to the quotient $(1 - e^{-i\omega\mu})/(1 - e^{-i\omega\nu})$, we write

$$F(z,z') = \frac{1}{2\pi}\int_{-\infty}^{\infty} \left[P_{\mu/\nu}{}^*(\omega,z) - \frac{1 - e^{i\omega\mu}}{1 - e^{i\omega\nu}}\, e^{-i\omega\mu}\right] \times \left[P_{\mu/\nu}(\omega,z') - \frac{1 - e^{-i\omega\mu}}{1 - e^{-i\omega\nu}}\, e^{i\omega\mu}\right] S(\omega,\nu;z,z')d\omega \quad (43)$$

An upper bound for the value of $\langle e^2\rangle$ may be readily obtained from Eqs. (38) and (42) by assuming that, at each level, the predicted value $u^p(t_1 + \nu,z)$ of the wind velocity at firing time $t_1 + \nu$ is taken equal to the wind velocity $u(t_1,z)$ at time t_1. This is equivalent to assuming a zero-prediction for the velocity increments. Thus, we make $P_\nu = P_\nu{}^* = 0$ in Eq. (42) and write

$$F(z,z') = \frac{1}{2\pi}\int_{-\infty}^{\infty} S(\omega,\mu;z,z')d\omega = D(0,\nu;z,z') \quad (44)$$

Substituting for $F(z,z')$ from (44) into (38), we obtain the upper bound of the mean square of the error:

$$\langle e^2\rangle = \int_0^{z_B}\int_0^{z_B} h(z,z_B)h(z',z_B)D(0,\nu;z,z')dzdz' \quad (45)$$

This result could have been obtained directly from Eq. (32).[7]

Returning to Eqs. (38, 42, and 43) which define the mean square $\langle e^2\rangle$ of the error, we observe that $\langle e^2\rangle$ depends upon the response $h(z,z_B)$ of the rocket to a unit impulse and upon the function $F(z,z')$. This function depends itself upon the various power spectra and cross-power spectra of the increments in time of the wind-velocity field. As noted in the preceding section, the spectra are independent of the mean values $c(z)\nu$ of the velocity increments and, thus, of any linear trend in the wind-velocity field. This observation holds also with regard to the cross-spectra. Both spectra and cross-spectra, however, depend upon the mean squares $\sigma_\nu{}^2(z)$ of the velocity increments $\Delta_\nu u(t,z)$. A formulation independent of the mean squares and, thus, valid under very general atmospheric conditions, may be obtained by normalizing all spectra. Setting

$$S(\omega,\nu;z,z') = \sigma_\nu(z)\sigma_\nu(z')S^N(\omega,\nu;z,z') \quad (46)$$

we write the mean square $\langle e^2\rangle$ of the error in the form

$$\langle e^2\rangle = \int_0^{z_B}\int_0^{z_B} h(z,z_B)h(z',z_B)\sigma_\nu(z)\sigma_\nu(z')F^N(z,z')dzdz' \quad (47)$$

where $\sigma_\nu(z)$ represents the root-mean square of the wind-velocity increments at elevation z, and where the function

$F^N(z,z')$ is obtained from the normalized spectra and cross-spectra through either of the relations (42) and (43).

VI. Conclusion

It was shown that the angle at burnout and the impact point of a free rocket fired at time $t_1 + \mu$ may be predicted from continuous wind-velocity records taken at various elevations z for $t \leqq t_1$. Wind-velocity increments $\Delta_\nu u(z,t)$ corresponding to a time interval ν are obtained from these records and operated upon by predictors whose transfer functions have been determined beforehand from the structure function of the wind-velocity field. The mean square value of the error in prediction of the angle at burnout, and thus, of the impact point, was shown to depend upon the cross-structure function of the same field. The determination of both functions involves the statistics of the velocity increments only and, thus, does not require the identification and elimination of the linear trend that is usually present in the wind-velocity field. The only statistical properties of the field which must be determined in the period $t \leqq t_1$ immediately before firing are the mean and the mean square of the velocity increments $\Delta_\nu u(t,z)$. The first value is required to correct the predictors' outputs, whereas the second is used in the computation of the mean square of the error.

Although there is evidence that the wind-velocity field may be assumed to possess stationary first increments, it should be noted that any second-order trend could be eliminated by considering second increments. As indicated earlier, the proposed extrapolation method may be readily extended to the case of higher-order stationary increments. The expressions obtained for the mean square $\langle e^2 \rangle$ of the error in prediction also remain valid in the case of increments of order n, except that in Eq. (43) the quantities $1 - e^{i\omega\mu}$, $1 - e^{i\omega\nu}$, and their conjugates must be replaced, respectively, by their nth powers.

A limitation and a possible refinement of the proposed method for the prediction of wind velocities at various levels should be indicated. Actual computation of predicted wind velocities has shown that considerable numerical errors are introduced for large values of the time interval μ. Although the proposed method, based on the prediction of wind-velocity increments, yields more accurate results for small values of μ than the method based on the direct prediction of the wind velocity, it appears that the latter (after removal of the linear trend) might be preferable for larger values of μ. Both methods, of course, are equally limited with regard to the theoretical value of the mean square of the error in prediction. It should also be pointed out that, with the proposed method, the prediction of the wind velocity at a given level is based entirely on the record of wind-velocity measurements at that level. A more refined method of prediction might take into account wind-velocity measurements at other levels.

References

[1] Bendat, J. S., Enochson, L. D., Klein, G. H., and Piersol, A. G., "Advanced concepts of stochastic processes and statistics for flight vehicle vibration estimation and measurement," ASD-TRD-62-973, Flight Dynamics Lab. Aeronautical Systems Div., Wright Patterson Air Force Base (December 1962).

[2] Wiener, N., *Extrapolation, Interpolation and Smoothing of Stationary Time Series* (John Wiley and Sons, Inc., New York, 1949).

[3] Yaglom, A. M., "Correlation theory of processes with random stationary *nth* increments," Am. Math. Soc. Transl. **8,** 87–141 (1958).

[4] Yaglom, A. M., *An Introduction to the Theory of Stationary Random Functions,* translated and edited by R. A. Silverman (Prentice Hall, Inc., Englewood Cliffs, N. J., 1962).

[5] Tatarskii, V. I., *Wave Propagation in a Turbulent Medium* (McGraw-Hill Book Co., New York, 1961).

[6] Bode, H. W. and Shannon, C. E., "A simplified derivation of linear least-square smoothing and prediction theory," Proc. Inst. Radio Engrs. **38,** 417–426 (1950).

[7] Beer, F. P. and Sarubbi, R. G., "The determination of the atmospheric parameters influencing the trajectory of a free rocket," Lehigh Univ. Institute of Research Rept. (August 1962).